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On the Continuity of the SRB Entropy for Endomorphisms*

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We consider classes of dynamical systems admitting Markov induced maps. Under general assumptions, which in particular guarantee the existence of SRB measures, we prove that the entropy of the SRB measure varies continuously with the dynamics. We apply our result to a vast class of non-uniformly expanding maps of a compact manifold and prove the continuity of the entropy of the SRB measure. In particular, we show that the SRB entropy of Viana maps varies continuously with the map.

KEY WORDS: SRB measures, entropy, induced maps, non-uniform expansion
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1. INTRODUCTION

In this work we address ourselves to the study of the continuity of the metric entropy for endomorphisms. Entropy of dynamical systems can be regarded quite generally as a measure of unpredictability. Topological entropy measures the complexity of a dynamical system in terms of the exponential growth rate of the number of orbits which can be distinguished over long time intervals, within a fixed small precision. Kolmogrov-Sinai's metric entropy is an invariant which, roughly speaking, measures the complexity of the dynamical system in probabilistic terms with respect to a fixed invariant measure.

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Considering that the observable properties are, in a physical sense, the properties which hold on a positive volume measure set, one tries to verify the existence of invariant measures with “good" densities with respect to the volume measure. Let us explain this in more precise terms. We consider discrete-time systems, namely, iterates of smooth transformations $f : M \to M$ on a Riemannian manifold. We consider a probability measure defined by a volume form on M that we call *Lebesgue measure*. A Borel probability measure μ on M is said to be a *Sinai-Ruelle-Bowen (SRB) measure* or a *physical measure*, if there exists a positive Lebesgue measure subset of points $x \in M$ for which

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \quad \text{for every } \varphi \in C^0(M). \tag{1.1}$$

The set of points $x \in M$ for which (1) holds is called the *basin* of the SRB measure μ. Finding SRB measures for a given dynamical system may be a difficult task in general. By Birkhoff's ergodic theorem, one possible way to prove the existence of these physically relevant measures is to construct absolutely continuous invariant ergodic probabilities. This kind of measures is constructed in Ref. (4) for a vast class of diffeomorphisms and endomorphisms satisfying some weak hyperbolicity conditions. We also refer the reader to Ref. (21) for results about statistical properties of dynamical systems with some hyperbolicity.

Recently, there is an increasing emphasis on the study of the stability of the statistical properties of dynamical systems One natural formulation for this kind of stability corresponds to the continuous variation of the SRB measures. Another interesting question in this direction is to ask whether the entropy of the SRB measure varies continuously as a function of the dynamical system. The question of the continuity of the entropy (topological or metric) is an old issue, going back to the work of Newhouse[14], for example.

It is known that uniformly expanding C^2 maps of a compact manifold admit a unique SRB measure which is absolutely continuous with respect to Lebesgue measure and its density varies continuously in the L^1 norm. By means of this continuity and the entropy formula for these systems one easily obtains the continuity of the SRB entropy. For Axiom A diffeomorphisms the continuity of SRB measures and even more regularity is established in Refs. (13) and (19). The regularity of the SRB entropy for Axiom A flows is proved in Ref. (8). Analiticity of metric entropy for Anosov diffeomorphisms is proved in Ref. (16).

In this paper we present an abstract model and give sufficient conditions which imply the continuous variation of the SRB entropy in quite general families of maps, including maps with critical sets. Under the same hypotheses, the continuous variation of the SRB measures is proved in Ref. (2). It is important to remark that in the presence of critical points it is not clear whether the continuous variation of absolutely continuous invariant measures implies the continuous variation of their entropy or not. Let us observe that if we do not have absolute continuity, the continuous variation of the SRB measures does not imply the continuity of their entropy. For instance, in the quadratic family $f_a(x) = 4ax(1-x)$ one can find parameters a for which f_a has an absolutely continuous SRB measure, and there is a sequence a_n converging to a with f_{a_n} having a unique SRB measure concentrated on an attracting periodic orbit (sink). Furthermore, the Dirac measures supported on those sinks converge to the SRB measure of f_a. This shows that the convergence of SRB measures does not necessarily imply the convergence of the SRB entropy.

In the sequel we show that a large class of *non-uniformly expanding* endomorphisms (admitting critical sets) satisfy the conditions of our main result. We just suppose some natural *slow recurrence* to the critical set to construct the absolutely continuous invariant measures as in Ref. (4). We apply our results to an open set of non-uniformly expanding endomorphisms constructed by Viana[20], and prove the continuity of the entropy of the unique absolutely continuous invariant measure for such endomorphisms.

As far as we know our result is the first one giving continuity of the SRB entropy for families of endomorphisms admitting critical points. Our approach is different from the usual ways to prove the continuity of the entropy. We construct induced maps for endomorphisms and relate the entropy of the SRB measure of the initial system and the entropy of a corresponding measure of the induced system. Then we prove some continuity results for the induced map and come back to the original map.

2. STATEMENT OF RESULTS

Let M be a d-dimensional compact Riemannian manifold and denote the Lebesgue measure on M by m. We are interested in studying the continuity of the metric entropy of smooth maps $f\colon M \to M$ with respect to some physically relevant measure on M.

A very important tool that we will be using are *induced maps*. Roughly speaking, an induced map for a system f is a transformation F from some region of the ambient space into itself, defined for each point as an iterate of f, where the number of iterations depends on the point. If we carry out this process carefully, some asymptotic properties of f (asymptotic expansion, for instance) can be verified as properties of F at the first iteration (real expansion) for almost all points. A hard problem is to decode back the information obtained for F into information about the original dynamical system.

2.1. Induced Maps

Let $F : \Delta \to \Delta$ be an *induced map* for f defined in some topological disk $\Delta \subset M$, meanning that there exists a countable partition $\mathcal{P}$ of a full Lebesgue measure subset of Δ, and there exists a *return time* function $\tau:\mathcal{P}\to\mathbb{Z}^+$ such that

$$F|_\omega = f^{\tau(\omega)}|_\omega, \qquad \text{for each} \qquad \omega\in\mathcal{P}.$$

We assume that the following conditions on the induced map F hold:

(i_1) Markov: $F|_\omega:\omega\to\Delta$ *is a* C^2 *diffeomorphism, for each* $\omega\in\mathcal{P}$.

(i_2) Uniform expansion: *there exists* $0<\kappa<1$ *such that for any* $\omega\in\mathcal{P}$ *and* $x\in\omega$

$$\|DF(x)^{-1}\|<\kappa.$$

(i_3) Bounded distortion: *there exists* $K>0$ *such that for any* $\omega\in\mathcal{P}$ *and* $x,y\in\omega$

$$\left|\frac{\det DF(x)}{\det DF(y)}-1\right|\le K\,\mathrm{dist}(F(x),F(y)).$$

It is well known that a map F in these conditions has a unique absolutely continuous ergodic invariant probability measure. Moreover, such a probability measure is equivalent to the Lebesgue measure on Δ, and its density is bounded from above and from below by constants. Proofs of these assertions will be given in Proposition 3.1. In this setting, we also prove in Proposition 4.3 that if $F:\Delta\to\Delta$ is a piecewise expanding Markov induced map and μ_F is its absolutely continuous invariant probability measure, then the entropy of F with respect to the probability measure μ_F satisfies:

$$h_{\mu_F}(F)=\int_\Delta \log|\det DF(x)|\,d\mu_F. \tag{2.1}$$

A natural question is how to obtain an absolutely continuous f-invariant probability measure from the existence of such measure for F. The integrability of the return time function $\tau:\Delta\to\mathbb{Z}^+$ with respect to the Lebesgue measure on Δ is enough for the existence of this measure. Indeed, if μ_F is the absolutely continuous F-invariant probability measure, then

$$\mu_f^*=\sum_{j=0}^{\infty} f_*^j(\mu_F\mid\{\tau_f>j\}) \tag{2.2}$$

is an absolutely continuous f-invariant finite measure. We denote by μ_f the probability measure which is obtained from μ_f^* by dividing it by its mass. Throughout this paper we are assuming the integrability of the return time.

A formula similar to the one displayed in (2.1) holds for C^2 endomorphisms f of a compact manifold M with respect to an absolutely continuous invariant probability measure μ_f. In fact, by (10, Remark 1.2) the Jacobian function $\log|\det Df(x)|$ is always integrable with respect to μ_f. Then, by (Ref. (17), Theorom 1.1) if

$$\lambda_1(x) \leq \cdots \leq \lambda_s(x) \leq 0 < \lambda_{s+1}(x) \leq \cdots \leq \lambda_d(x)$$

are the Lyapunov exponents at x, then

$$h_{\mu_f}(f) = \int_M \sum_{i=s+1}^{d} \lambda_i(x)\, d\mu_f(x). \tag{2.3}$$

We will refer to this last equality as the *entropy formula* for μ_f. We will see in Lemma 4.1 that in our situation f has all its Lyapunov exponents positive with respect to μ_f. Hence, by Oseledets Theorem and the integrability of the Jacobian of f with respect to μ_f, we have that the integral in (2.3) is equal to the integral of the Jacobian of f with respect to the measure μ_f; see Proposition 4.2.

One of the key results to prove our main result on continuity of the SRB entropy is the following well-know Theorem (see Ref. (9), p. 254) which establishes the relation between the entropy of the original map and the entropy of the induced map with respect to the appropriate measures.

Theorem A. *If F is an induced map for f and μ_f and μ_F are related as in (3), then*

$$h_{\mu_f}(f) = \frac{1}{\mu_f^*(M)}\, h_{\mu_F}(F).$$

For the sake of completeness, a proof of this result will be given in Section 4.

2.2. Continuity of Entropy

Let $\mathcal{U}$ be a family of C^k maps, for some fixed $k \geq 2$, from a manifold M into itself. Assume that we may associate to each $f \in \mathcal{U}$ an induced Markov map $F_f\colon \Delta \to \Delta$ defined on a ball $\Delta \subset M$ that do not depends on $f \in \mathcal{U}$. Given $f \in \mathcal{U}$, let $\mathcal{P}_f$ denote the partition into domains of smoothness of F_f, and that its return time function $\tau_f\colon \mathcal{P}_f \to \mathbb{Z}^+$ is integrable. Let also μ_{F_f} be the absolutely continuous F_f-invariant probability measure, μ_f^* the measure obtained from μ_{F_f} as in (2.2), and μ_f its normalization. For notational simplicity we will denote the Markov induced map associated to f by F and its absolutely continuous invariant probability measure by μ_F.

One of the main goals of this work is to study the continuous variation of the metric entropy with respect to μ_f with the map $f \in \mathcal{U}$. In order to be able to implement our strategy we assume that the following uniformity conditions hold:

(u$_1$) *τ_f varies continuously in the L^1 norm (wrt Lebesgue measure) with $f \in \mathcal{U}$.*
(u$_2$) *κ and K associated to F_f as in (i$_2$) and (i$_3$) may be chosen uniformly for $f \in \mathcal{U}$.*

The results in (6) give that under these uniformity assumptions the measure μ_f^* varies continuously in the L^1 norm with $f \in \mathcal{N}$. As we shall see in Proposition 3.3, these uniformity conditions assure also that the (unique) absolutely continuous probability measure μ_F invariant by the map F varies continuously (in the L^1 norm) with $f \in \mathcal{U}$.

Theorem B. *If $\mathcal{U}$ is a family of C^k ($k \geq 2$) maps from the manifold M into itself for which (u$_1$) and (u$_2$) hold, then the entropy $h_{\mu_f}(f)$ varies continuously with $f \in \mathcal{U}$.*

Next we introduce a family of maps and present sufficient conditions for the validity of the assumptions of the previous theorem. As we shall see these conditions are verified in the set of maps introduced in Ref. (20).

2.3. Non-uniformly Expanding Maps

Let $f\colon M \to M$ be a C^2 local diffeomorphism in the whole manifold M except possibly in a set of critical points $\mathcal{C} \subset M$. We say that $\mathcal{C}$ is a *non-degenerate critical set* if the following conditions hold. The first one says that there are constants $B > 0$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{C}$ one has

$$(c_1) \qquad \|Df(x)\| \geq B\,\mathrm{dist}(x, \mathcal{C})^{\beta}.$$

Moreover, we assume that the functions $\log|\det Df|$ and $\log\|Df^{-1}\|$ are *locally Lipschitz* at points $x \in M \setminus \mathcal{C}$, with Lipschitz constant depending on $\mathrm{dist}(x, \mathcal{C})$: for every $x, y \in M \setminus \mathcal{C}$ with $\mathrm{dist}(x, y) < \mathrm{dist}(x, \mathcal{C})/2$ we have

$$(c_2) \qquad |\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|| \leq \frac{B}{\mathrm{dist}(x, \mathcal{C})^{\beta}}\mathrm{dist}(x, y);$$

$$(c_3) \qquad |\log|\det Df(x)^{-1}| - \log|\det Df(y)^{-1}|| \leq \frac{B}{\mathrm{dist}(x, \mathcal{C})^{\beta}}\mathrm{dist}(x, y).$$

Note that the above conditions give, for the particular case of critical points of one-dimensional maps, the usual definition of a non-degenerate critical point. From now on we assume that the critical sets of the maps we will be considering are always non-degenerate.

Given any $\delta > 0$ and $x \in M \setminus \mathcal{C}$, we define the *$\delta$-truncated distance* from x to $\mathcal{C}$ as

$$\mathrm{dist}_\delta(x, \mathcal{C}) = \begin{cases} 1, & \text{if } \mathrm{dist}(x, \mathcal{C}) \geq \delta; \\ \mathrm{dist}(x, \mathcal{C}), & \text{otherwise.} \end{cases}$$

We say that f is *non-uniformly expanding* if the following two conditions hold:

(n_1) *there is* $\lambda > 0$ *such that for Lebesgue almost every* $x \in M$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda;$$

(n_2) *for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for Lebesgue almost every* $x \in M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \mathrm{dist}_\delta(f^j(x), \mathcal{C}) \le \epsilon.$$

We often refer to (n_2) by saying that orbits have *slow recurrence* to the critical set $\mathcal{C}$. In the case that $\mathcal{C}$ is equal to the empty set we simply ignore the slow recurrence condition.

Remark 2.1. *It is worthy to be stressed that slow recurrence condition is not needed in all its strength for our results. In fact, condition (n_2) is needed just for distortion control reasons. As observed in (2, Remark 1.3), it is enough to have it for some sufficiently small* $\epsilon > 0$ *and conveniently chosen* $\delta > 0$*; see also (Ref. (2), Proposition 3.5) and (Ref. (2), Remark 3.6).*

Condition (n_1) implies that the *expansion time* function

$$\mathcal{E}(x) = \min\left\{ N \ge 1\colon \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -\frac{\lambda}{2}, \quad \text{for all } n \ge N \right\}$$

is defined and finite Lebesgue almost everywhere in M. The *recurrence time* function

$$\mathcal{R}(x) = \min\left\{ N \ge 1 : \frac{1}{n} \sum_{j=0}^{n-1} -\log \mathrm{dist}_\delta(f^j(x), \mathcal{C}) \le 2\varepsilon, \quad \text{for all } n \ge N \right\},$$

is also defined and finite Lebesgue almost everywhere in M if the slow recurrence condition (n_2) holds.

We think of $\mathcal{E}(x)$ and $\mathcal{R}(x)$ as the time we need to wait before the exponential derivative growth kicks in. Note that $\mathcal{E}(x)$ and $\mathcal{R}(x)$ depends on suitable choice of the constants λ, ϵ and δ. In our applications, these constants will be taken fixed once for all and for sake of clearness we will supress it. These numbers also depend on asymptotic statements and we have no a-priori knowledge about how fast these limits are approached or with what degree of uniformity for different points x. We define the *tail set (at time n)*

$$\Gamma_n^f = \{x \in M\colon \mathcal{E}(x) > n \quad \text{or} \quad \mathcal{R}(x) > n\}. \tag{2.4}$$

This is the set of points which at time n have not yet achieved either the uniform exponential growth or the slow recurrence given by conditions (n_1) and (n_2). If the critical set is empty, we simply ignore the recurrence time function and consider only the expansion time function in the definition of Γ_n^f.

It is proved in Ref. (4) that every C^2 non-uniformly expanding map f admits some SRB measure. Moreover, it follows from (Ref. (4), Lemma 5.6) that if f is transitive, then it has a unique SRB measure μ_f which is ergodic and absolutely continuous with respect to the Lebesgue measure, whose basin covers a full Lebesgue measure subset of points in M.

The results in (2) show that if the decay of the Lebesgue measure of Γ_n^f holds with some uniformity in $f \in \mathcal{N}$, then the SRB measure μ_f varies continuously in the L^1 norm with $f \in \mathcal{N}$. Here we deduce the continuity of the SRB entropy in the same context.

Theorem C. *Let $\mathcal{N}$ be a family of C^k ($k \geq 2$) transitive non-uniformly expanding maps (with same constants ϵ, δ and λ). If there are $C > 0$ and $\gamma > 1$ such that $\mathrm{Leb}(\Gamma_n^f) \leq Cn^{-\gamma}$, for all $f \in \mathcal{N}$ and $n \geq 1$, then the entropy $h_{\mu_f}(f)$ varies continuously with $f \in \mathcal{N}$.*

Using results from C one can prove that maps in a family $\mathcal{N}$ as in the hypotheses of Theorem C necessarily admit induced maps for which uniformity conditions (u_1) and (u_2) hold. The proof of the results in (2) uses ideas from (5), where induced piecewise expanding maps for non-uniformly expanding maps are constructed. Transitivity is an important ingredient for that construction.

2.3.1. Viana Maps

Here we present an open class $\mathcal{V}$ of transformations where the assumptions of Corollary C hold. This is an open set of maps from the cylinder into itself constructed in (20). As pointed out in that paper, the choice of the cylinder $S^1 \times \mathbb{R}$ as ambient space is rather arbitrary, and the construction extends easily to more general manifolds. In what follows we briefly describe the maps in the set $\mathcal{V}$, and refer the reader to (1, 3, 6, 7, 20) for more details.

Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic under iteration by the quadratic map $p(x) = a_0 - x^2$, and let $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(t) = \sin(2\pi t)$. We take $S^1 = \mathbb{R}/\mathbb{Z}$. For each $\alpha > 0$, consider the map

$$f_\alpha : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad f_\alpha(\theta, x) = (\hat{g}(\theta), \hat{q}(\theta, x)),$$

where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(\theta) = d\theta$ (mod 1), for some integer $d \geq 2$, and $\hat{q}(\theta, x) = a(\theta) - x^2$ with $a(\theta) = a_0 + \alpha b(\theta)$. We take $\mathcal{V}$ as a small C^3 neighborhood of f_α, for some (fixed) sufficiently small $\alpha > 0$.

Observe that each $f \in \mathcal{V}$ has a whole curve of critical points near $\{x = 0\}$ for small enough $\alpha > 0$. The C^3 topology is used in (20) in order to simplify some technical points. In particular, it is possible to prove C^2 proximity of the critical sets for C^3 nearby maps. We do believe that the results in Ref. (20) and the subsequent works for Viana maps still hold in the C^2 topology.

One can easily check that for $\alpha > 0$ small enough there is an interval $I \subset (-2, 2)$ such that f_α sends $S^1 \times I$ into the interior of $S^1 \times I$. Thus, any map f close to f_α still has $S^1 \times I$ as a forward invariant region, and so it has an attractor inside this invariant region. The attractor is precisely the set $\Lambda = \cap_{n \geq 0} f^n(S^1 \times I)$.

It is proved in (1) that any $f \in \mathcal{V}$ admits some absolutely continuous ergodic invariant probability measure. Moreover, the results in (6) show that these systems have a unique SRB measure whose basin covers a full Lebesgue measure set of points in $S^1 \times I$, and the densities of these SRB measures vary continuously in the L^1 norm with the map. To obtain the uniqueness of the SRB measure, they prove that f is *topologically mixing*, in a strong sense: for every open set $A \subset S^1 \times I$ there is some $n = n(A) \in \mathbb{Z}^+$ such that $f^n(A) = \Lambda$. In particular, maps belonging to $\mathcal{V}$ are transitive.

The non-uniform expansivity of Viana maps is proved in Ref. (20). Specific rates for the decay of the tail set are known in this case: there exist constants $C, \gamma > 0$ (uniformly in the whole set $\mathcal{V}$) such that

$$m\left(\Gamma_n^f\right) \leq C \exp(-\gamma\sqrt{n}), \quad \text{for all } f \in \mathcal{V} \text{ and } n \geq 1;$$

see (Ref. (20), Section 2.4) and (Ref. (3), Section 6.2) for details. Thus we may apply Corollary C tmo the set of Viana maps and derive the following consequence.

Corollary D. *The SRB entropy of Viana maps varies continuously with $f \in \mathcal{V}$.*

Let us remark that $\mathcal{V}$ is an open set in the space of C^3 transformations from the cylinder $S^1 \times I$ into itself, where each $f \in \mathcal{V}$ has a curve of critical points. The conclusion on the continuity of the SRB entropy in this higher dimensional case is completely different from the above mentioned case of one-dimensional quadratic maps.

3. STATISTICAL STABILITY

Let $\mathcal{U}$ be a family os maps as in Theorem B. The main goal of this section is to prove that μ_F varies continuously with $f \in \mathcal{N}$. In the next lemma we give in particular a proof that an absolutely continuous invariant measure for a piecewise expanding Markov map exists. For the sake of notational simplicity we shall write

$$J_f(x) = |\det Df(x)| \quad \text{and} \quad J_F(x) = |\det DF(x)|.$$

Proposition 3.1. *There is $C_0 > 0$ such that for each $f \in \mathcal{U}$ there exists an F-invariant absolutely continuous probability measure $\mu_F = \rho_F m$ with $C_0^{-1} \leq \rho_F \leq C_0$.*

Proof. We start the proof of the result with the following claim: there exists $K_0 > 0$ such that given $f \in \mathcal{U}$, $k \geq 1$, an inverse branch $G : \Delta \to G(\Delta)$ of F^{-k}, and measurable sets $A, B \subset \Delta$, then

$$K_0^{-1}\frac{m(A)}{m(B)} \leq \frac{m(G(A))}{m(G(B))} \leq K_0\frac{m(A)}{m(B)}. \tag{3.1}$$

Indeed, observe that

$$\frac{m(A)}{m(B)} = \frac{\int_{G(A)} J_{F^k}dm}{\int_{G(B)} J_{F^k}dm}$$

We use (i_3) and show that there is $K_1 > 0$ (uniformly choosen in $\mathcal{U}$) such that

$$K_1^{-1} \leq \frac{J_{F^k}(y)}{J_{F^k}(z)} \leq K_1 \tag{3.2}$$

for every y, z on the image of G. For this purpose observe that

$$\begin{aligned}\log\frac{J_{F^k}(y)}{J_{F^k}(z)} &= \sum_{i=0}^{k-1}\log\frac{J_F(F^i(y))}{J_F(F^i(z))}\\ &\leq \sum_{i=0}^{k-1}\left|\frac{J_F(F^i(y))}{J_F(F^i(z))} - 1\right|\\ &\leq K\sum_{i=1}^{k}\operatorname{dist}(F^i(y), F^i(z))\\ &\leq K\sum_{i=1}^{\infty}\kappa^i L,\end{aligned}$$

where L is the diameter of M. Observe that the last upper bound is uniform in $\mathcal{U}$.

Now we use (3.2) to prove (3.1). Fixing $z \in G(\Delta)$, it comes out that

$$\frac{\int_{G(A)} J_{F^k}dm}{\int_{G(B)} J_{F^k}dm} \leq K_1^2\frac{J_{F^k}(z)m(G(A))}{J_{F^k}(z)m(G(B))},$$

and with the same argument we prove the other inequality of (3.1) with $K_1^2 = K_0$.

Using the claim we will see that every accumulation point μ_F of the sequence

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} F_*^i m$$

is an F-invariant probability absolutely continuous with respect to m, with density ρ_F bounded from zero and from infinity. In order to prove it, take $B = \Delta$ and fix $C_0 = K_0 m(\Delta)^{-1}$. Since $m(F^{-k}(A))$ is the sum of the terms $m(G(A))$ over all inverse branches $G : \Delta \to G(\Delta)$ of F^k, it follows from (6) that

$$C_0^{-1} m(A) \le m(F^{-k}(A)) \le C_0 m(A).$$

This implies that, for every n, the density $\rho_n = d\mu_n/dm$ satisfies $C_0^{-1} \le \rho_n \le C_0$, and the same holds for the density of the accumulation point μ_F. □

Lemma 3.2. *Given $\epsilon > 0$, there are $N \ge 1$ and $\delta = \delta(\epsilon, N) > 0$ such that for $f \in \mathcal{U}$*

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad m\{\tau_f > N\} < \varepsilon.$$

Proof. For the sake of notational simplicity we denote τ_f by τ and τ_{f_0} by τ_0. Take any $\epsilon > 0$ and take $N \ge 1$ in such a way that $\|\mathbf{1}_{\{\tau_0 > N\}}\|_1 < \varepsilon/2$, where $\mathbf{1}_A$ denotes the indicator of a set A. We have

$$\begin{aligned} m\{\tau > N\} &= \left\|\mathbf{1}_{\{\tau > N\}}\right\|_1 \\ &= \left\|\mathbf{1}_{\{\tau > N\}} - \mathbf{1}_{\{\tau_0 > N\}} + \mathbf{1}_{\{\tau_0 > N\}}\right\|_1 \\ &\le \left\|\mathbf{1}_{\{\tau > N\}} - \mathbf{1}_{\{\tau_0 > N\}}\right\|_1 + \left\|\mathbf{1}_{\{\tau_0 > N\}}\right\|_1 \end{aligned}$$

and so, if we take $\delta > 0$ sufficiently small then, by (u_1), taking $\|f - f_0\|_{C^k} < \delta$, the first term in the sum above can also be made smaller than $\varepsilon/2$. □

Proposition 3.3. *The density of measure μ_F varies continuously (in the weak-* topology of L^∞) with $f \in \mathcal{U}$.*

Proof. Let f_n be any sequence in $\mathcal{U}$ converging to f_0 in the C^k topology. For each $n \ge 0$, consider $F_n : \Delta \to \Delta$ the induced Markov map associated to f_n. Denote by ρ_n the density of the F_n-invariant absolutely continuous probability measure. Proposition 3.1 gives that the sequence of densities ρ_n belongs to some ball in $L^\infty(\Delta, m)$, and so, by Banach-Alaogulu Theorem, it has some accumulation point ρ_∞ in this ball with respect to the weak* topology. This means that for every $\phi \in L^1(\Delta, m)$ we have $\int \phi\rho_n$ converging to $\int \phi\rho_\infty$ as $n \to \infty$, and $\|\rho_\infty\|_\infty \le C_0$, as in Proposition 3.1. With no loss of generality we assume that the full sequence

ρ_n converges to ρ_∞. We need to prove that $\rho_\infty = \rho_0$. We will do this by showing that

$$\int (\varphi \circ F_0)\rho_\infty dm = \int \varphi\rho_\infty dm$$

for every continuous $\varphi\colon \Delta \to \mathbb{R}$, and use the fact that F_0 has a unique absolutely continuous invariant probability measure. Given any $\varphi\colon M \to \mathbb{R}$ continuous we have

$$\int \varphi\rho_n dm \to \int \varphi\rho_\infty dm \quad \text{when} \quad n \to \infty.$$

On the other hand, since ρ_n is the density of an F_n-invariant probability measure we have

$$\int \varphi\rho_n dm = \int (\varphi \circ F_n)\rho_n dm \quad \text{for every } n \geq 0.$$

So, it suffices to prove that

$$\int (\varphi \circ F_n)\rho_n dm \to \int (\varphi \circ F_0)\rho_\infty dm \quad \text{when} \quad n \to \infty. \tag{3.3}$$

We have

$$\left|\int (\varphi \circ F_n)\rho_n dm - \int (\varphi \circ F_0)\rho_\infty dm\right| \leq$$
$$\left|\int (\varphi \circ F_n)\rho_n dm - \int (\varphi \circ F_0)\rho_n dm\right| + \left|\int (\varphi \circ F_0)\rho_n dm - \int (\varphi \circ F_0)\rho_\infty dm\right|.$$

Observing that $\varphi \circ F_0$ is bounded, thus integrable since m is finite, we easily deduce from the convergence of ρ_n to ρ_∞ in the weak* topology that the second term in the sum above is close to zero for large n.

The only thing we are left to prove is that the first term in the sum above converges to 0 when n tends to ∞. That term is equal to

$$\left|\int (\varphi \circ F_n - \varphi \circ F_0)\rho_n dm\right|.$$

Since $(\rho_n)_n$ is bounded in the L^∞ norm by Proposition 3.1, all we are left to show is

$$\int |\varphi \circ F_n - \varphi \circ F_0| dm \to 0, \quad \text{when} \quad n \to \infty. \tag{3.4}$$

Take any $\varepsilon > 0$. For each $n \geq 0$ let τ_n denote the return time function of f_n. By Lemma 3.2 there are $N \geq 1$ and $n_1 \in \mathbb{N}$ such that

$$n \geq n_1 \quad \Rightarrow \quad m(\{\tau_n > N\}) < \varepsilon.$$

We write the integral in 3.4 as

$$\int_{\{\tau_n>N\}} |\varphi\circ F_n-\varphi\circ F_0|dm+\int_{\{\tau_n\le N\}} \big|\varphi\circ F_n-\varphi\circ F_0\big|dm. \tag{3.5}$$

The first integral in (3.5) is bounded by $2\varepsilon\|\varphi\|_\infty$ for $n\ge n_1$. Let us now estimate the second integral in (3.5). Define

$$A_n=\big\{x\in\Delta\colon \tau_n(x)=\tau_0(x)\big\}.$$

Since τ_n takes only integer values, we have by (u_1) that there is some $n_2\in\mathbb{N}$ such that

$$m(\Delta\setminus A_n)\le\varepsilon,\qquad \text{for each}\, n\ge n_2.$$

Observe that for each $x\in A_n$ we have $F_n(x)=f_n^{\tau_0(x)}(x)$. Thus we may write

$$\int_{\{\tau_n\le N\}} |\varphi\circ F_n-\varphi\circ F_0|dm\le\int_{\{\tau_0\le N\}} |\varphi\circ f_n^{\tau_0}-\varphi\circ f_0^{\tau_0}|dm$$
$$+\int_{\Delta\setminus A_n} |\varphi\circ F_n-\varphi\circ F_0|dm.$$

Since $f_n\to f_0$ in the C^k topology, there is $n_3\in\mathbb{N}$ such that for $n\ge n_3$

$$\int_{\{\tau_0\le N\}} |\varphi\circ f_n^{\tau_0}-\varphi\circ f_0^{\tau_0}|dm\le\varepsilon m(\{\tau_0\le N\}).$$

On the other hand, for $n\ge n_2$

$$\int_{\Delta\setminus A_n} |\varphi\circ F_n-\varphi\circ F_0|dm\le 2\varepsilon\|\varphi\|_\infty.$$

Thus we have for $n\ge\max\{n_1,n_2,n_3\}$

$$\int |\varphi\circ F_n-\varphi\circ F_0|dm\le\varepsilon(4\|\varphi\|_\infty+m(\{\tau_0\le N\})).$$

This proves (3.4) since $\varepsilon>0$ has been taken arbitrarily. □

4. ENTROPY FORMULAS

In this section we prove Theorem 0. Let $F\colon\Delta\to\Delta$ be a piecewise expanding Markov map and μ_F its absolutely continuous invariant probability measure. Since the Lypunov exponents of the induced map F (with respect to the measure μ_F) are all positive, then the next lemma shows, in particular, that the Lyapunov exponents of f (with respect to the measure μ_f) are all positive.

Lemma 4.1. *If λ is a Lyapunov exponent of F, then $\lambda/\bar\tau$ is a Lyapunov exponent of f, where $\bar\tau=\int_\Delta \tau_f d\mu_F$.*

Proof. Let n be a positive integer. We have for each $x \in \Delta$

$$F^n(x) = f^{S_n(x)}(x), \quad \text{where } S_n(x) = \sum_{i=0}^{n-1} \tau_f(F^i(x)).$$

As $S_n(x) = S_n(y)$ for Lebesgue almost every $x \in \Delta$ and y near enough x, we can take derivatives in the above equation and conclude that if $v \in T_x M$ then

$$\frac{1}{S_n(x)} \log \|Df^{S_n(x)}(x)v\| = \frac{n}{nS_n(x)} \log \|DF^n(x)v\|. \tag{4.1}$$

Since μ_F is an ergodic measure, we have by Birkhoff's ergodic theorem

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \int_\Delta \tau_f d\mu_F = \bar{\tau} \tag{4.2}$$

for Lebesgue almost every $x \in \Delta$ (recall that μ_F is equivalent to Lebesgue measure). Attending to (4.1) and (4.2) the proof follows. □

Proposition 4.2. *The entropy formula holds for μ_f, i.e. $h_{\mu_f}(f) = \int_M \log J_f \, d\mu_f$.*

Proof. As a consequence of Lemma 4.1, the fact that the Lyapunov exponents of F with respect to μ_F are all positive implies that all Lyapunov exponents of f with respect to μ_f are also positive. By the entropy formula

$$h_{\mu_f} = \int_M \sum_{i=1}^{d} \lambda_i d\mu_f.$$

Now the integrability of $\log Jf$ with respect to μ_f allows us to use Oseledets Theorem and rewrite the above equality as required in the above proposition. □

The proof of the next proposition uses fairly standard methods in ergodic theory.

Proposition 4.3. *If $F\colon \Delta \to \Delta$ is a piecewise expanding map for which (i_1), (i_2) and (i_3) hold, then*

$$h_{\mu_F}(F) = \int_\Delta \log J_F \, d\mu_F.$$

Proof. First we observe that the measure μ_F is ergodic. We shall apply Shannon-McMillan-Breiman theorem for the generating partition $\mathcal{P}$ consisting of the smoothness domains of F. The partition $\mathcal{P}$ is generating just because of i_2. We need to show that $H(\mathcal{P}) < \infty$. To show this, let

$$a_n := \mu_F(\{\tau = n\}).$$

So, by the integrability of τ, we have $\sum_{n=1}^{\infty} n a_n < \infty$. For $a_n > 1/n^2$ we have $a_n \log(a_n) < 2a_n \log(n)$. Using the fact that the function $-x\log(x)$ is increasing on an interval near zero, for $a_n \leq 1/n^2$ we have $a_n \log(a_n) \leq \frac{2\log(n)}{n^2}$. Finally

$$\begin{aligned} -\sum_{n=1}^{\infty} a_n \log a_n &\leq -\left(\sum_{\{n:a_n \leq 1/n^2\}} a_n \log a_n + \sum_{\{n:a_n > 1/n^2\}} a_n \log a_n \right) \\ &\leq \sum_{n=1}^{\infty} \frac{2\log n}{n^2} + \sum_{n=1}^{\infty} 2a_n \log n \\ &\leq \sum_{n=1}^{\infty} Cna_n < \infty. \end{aligned}$$

Take a generic point $x \in \Delta$. We have

$$h_{\mu_F}(F) = h_{\mu_F}(F, \mathcal{P}) = \lim_{n\to\infty} \frac{-1}{n} \log \mu_F(\mathcal{P}_n(x)) = \lim_{n\to\infty} \frac{-1}{n} \log m(\mathcal{P}_n(x)). \tag{4.3}$$

The last equality comes from the fact that m and μ_F are equivalent measures with uniformly bounded densities. Now observe that each $\mathcal{P}_n(x)$ is equal to some $G(\Delta)$, where G is an inverse branch of F^n. Hence we have

$$m(\Delta) = \int_{G(\Delta)} J_{F^n} dm. \tag{4.4}$$

By the distortion estimate obtained in the proof of the Proposition 3.1 we conclude that

$$K_1^{-1} \leq m(G(\Delta)) J_{F^n}(x) \leq K_1.$$

By the above inequality we deduce that

$$\begin{aligned} &\lim_{n\to\infty} \frac{-1}{n} \log m(\mathcal{P}_n(x)) = \lim_{n\to\infty} \frac{1}{n} \log J_{F^n}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log J_F(F^i(x)) \\ &= \int \log J_F d\mu_F, \end{aligned}$$

where the last equality holds by Birkhoff's ergodic theorem. □

Now we give a lemma with the aid of which we shall prove Theorem A.

Lemma 4.4. *If F is an induced piecewise expanding Markovian map for f, then*

$$\int_\Delta \log J_F \, d\mu_F = \int_M \log J_f \, d\mu_f^*.$$

Proof. We define for each $n \geq 1$

$$P_n = \{\omega \in \mathcal{P} : \tau(\omega) = n\}.$$

Observe that for each $x \in P_n$ we have $F = f^n$. So, by the chain rule,

$$J_F(x) = J_f(f^{n-1}(x)) \cdots J_f(f(x)) \cdot J_f(x).$$

Thus we have for each $n \geq 1$

$$\begin{aligned}
&\int_{P_n} \log J_F d\mu_F = \\
&= \int_{P_n} \log J_f \circ f^{n-1} d\mu_F + \cdots + \int_{P_n} \log J_f \circ f d\mu_F + \int_{P_n} \log J_f d\mu_F \\
&= \int_M \log J_f \, d\left(f_*^{n-1}(\mu_F|P_n)\right) + \cdots + \int_M \log J_f \, d\left(f_*(\mu_F|P_n)\right) \\
&\quad + \int_M \log J_f \, d(\mu_F|P_n).
\end{aligned}$$

Using this we deduce

$$\begin{aligned}
\int_\Delta \log J_F d\mu_F &= \sum_{n=1}^{\infty} \int_{P_n} \log J_F \, d\mu_F \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \int_M \log J_f \, d\left(f_*^j(\mu_F|P_n)\right) \\
&= \sum_{n=0}^{\infty} \int_M \log J_f \, d\left(f_*^n(\mu_F|\{\tau > n\})\right) \\
&= \int_M \log J_f \, d\left(\sum_{n=0}^{\infty} f_*^n(\mu_F|\{\tau > n\})\right).
\end{aligned}$$

By 2.2 we have

$$\int_M \log J_f \, d\left(\sum_{n=0}^{\infty} f_*^n(\mu_F|\{\tau > n\})\right) = \int_M \log J_f d\mu_f^*,$$

and so we have proved the result. □

Since the entropy formula holds for μ_f by Proposition 4.2, then using Proposition 4.3 and Lemma 4.4 we obtain

$$\begin{aligned} h_{\mu_f}(f) &= \int_M \log J_f \, d\mu_f \\ &= \frac{1}{\mu_f^*(M)} \int_M \log J_f \, d\mu_f^* \\ &= \frac{1}{\mu_f^*(M)} \int_\Delta \log J_F \, d\mu_F \\ &= \frac{1}{\mu_f^*(M)} h_{\mu_F}(F). \end{aligned} \tag{4.5}$$

This proves Theorem A.

5. CONTINUITY OF ENTROPY

In this section we prove Theorem B. Let $\mathcal{U}$ be a family of C^k maps, $k \geq 2$, from the manifold M into itself for which (u_1) and (u_2) hold. We are implicitly assuming that we have some $\Delta \subset M$ and, associated to each $f \in \mathcal{U}$, a piecewise expanding Markov induced map $F\colon \Delta \to \Delta$. By (4.5), in order to prove Theorem B, we just have to show that both $\mu_f^*(M)$ and $\int_\Delta \log J_F \, d\mu_F$ vary continuously with $f \in \mathcal{U}$.

Take an arbitrary $f_0 \in \mathcal{U}$ and let f_n be any sequence in $\mathcal{U}$ converging to f_0 in the C^k topology. For each $n \geq 0$, let $F_n\colon \Delta \to \Delta$ be the induced map associated to f_n, and let $\tau_n\colon \Delta \to \mathbb{N}$ be the respective return time function. Denote by ρ_n the density of the absolutely continuous F_n-invariant probability measure μ_{F_n}. Consider also for $n \geq 0$ the absolutely continuous f_n-invariant measure μ_n^* obtained as in (2.2) from μ_{F_n}:

$$\mu_n^* = \sum_{j=0}^{\infty} (f_n^j)_*(\mu_{F_n} \mid \{\tau_f > j\}).$$

The continuous variation of $\mu_f^*(M)$ and $\int_\Delta \log J_F \, d\mu_F$ with $f \in \mathcal{U}$ will follow from Propositions 5.2 and 5.4 below. We start with an abstract lemma.

Lemma 5.1. *Let $(\varphi_n)_n$ be a bounded sequence in $L^\infty(m)$. If $\varphi_n \to \varphi$ in the weak* topology of $L^\infty(m)$ and $\psi \in L^1(m)$, then*

$$\lim_{n\to\infty} \int \psi(\varphi_n - \varphi) dm = 0.$$

Proof. Take any $\epsilon > 0$. Let $C > 0$ be an upper bound for $\|\varphi_n\|_\infty$. Since $\psi \in L^1(m)$, there is N such that for $B_N := \{x \in \Delta : \psi(x) \leq N\}$ we have:

$$\int_{\Delta \setminus B_N} \psi dm \leq \epsilon. \tag{5.1}$$

Taking into account the definition of B_N, we may write

$$\begin{aligned} |\int \psi(\varphi_n - \varphi_0)dm| &\leq |\int_{B_N} \psi(\varphi_n - \varphi_0)dm| + |\int_{\Delta\setminus B_N} \psi(\varphi_n - \varphi_0)dm| \\ &\leq N|\int_{B_N} (\varphi_n - \varphi)dm| + 2C|\int_{\Delta\setminus B_N} \psi dm| \\ &\leq N|\int_{B_N} (\varphi_n - \varphi)dm| + 2C\epsilon. \end{aligned}$$

Observe that $\varphi_n \to^* \varphi$ implies that

$$\lim_{n\to\infty} |\int_{B_N} (\varphi_n - \varphi)dm| = 0$$

and as ϵ was arbitrary the proof is finished. □

Proposition 5.2. *$\mu_n^*(M)$ converges to $\mu_0^*(M)$ when $n \to \infty$.*

Proof. Recall that we have for every $n \geq 0$

$$\mu_n^*(M) = \sum_{j=0}^{\infty} \mu_{F_n}(\{\tau_n > j\}) = \int \tau_n d\mu_{F_n}.$$

Hence

$$|\mu_n^*(M) - \mu_0^*(M)| \leq \int |\tau_n\rho_n - \tau_0\rho_0|dm.$$

Now we write

$$\int |\tau_n\rho_n - \tau_0\rho_0|dm \leq \int |\tau_0||\rho_n - \rho_0|dm + \int |\tau_n - \tau_0||\rho_n|dm. \tag{5.2}$$

Let us first control the first term on the right hand side of (5.2). If we take $\psi = \tau_0$ and $\varphi_n = \rho_n$ for each $n \geq 0$, then, by Propositions 3.1 and 3.3, these functions are in the conditions of Lemma 5.1. Hence, the first term on the right hand side of (5.2) converges to 0 when $n \to \infty$. We just have to notice that

$$\int |\tau_n - \tau_0||\rho_n|dm \to 0, \quad \text{when} \quad n \to \infty. \tag{5.3}$$

In fact, since $(\rho_n)_n$ is uniformly bounded by Proposition 3.1, then hypothesis (u_1) assures that (5.3) holds. □

At this point we have proved the continuous variation of $\mu_f^*(M)$ with $f \in \mathcal{U}$, thus attaining the first step in the proof of Theorem B. The next step is to prove the continuous variation of $\int_\Delta \log J_F \, d\mu_F$ with $f \in \mathcal{U}$. We start with an auxiliary lemma.

Lemma 5.3. *There is* $C > 0$ *such that* $\log J_{F_n} \leq C\tau_n$ *for every* $n \geq 0$.

Proof. Define $K_n = \max_{x\in M}\{J_{f_n}(x)\}$, for each $n \geq 0$. By the compactness of M and the continuity on the first order derivative, there is $K > 1$ such that $K_n \leq K$ for all $n \geq 0$. We have

$$J_{F_n}(x) = \prod_{j=0}^{\tau_n(x)-1} J_{f_n}(f_n^j(x)) \leq K^{\tau_n(x)}.$$

Hence

$$0 < \log J_{F_n}(x) \leq \tau_n(x) \log K.$$

We just have to take $C = e^K$. □

The previous result gives in particular the integrability of $\log J_f$ with respect to Lebesgue measure, under the assumption of the integrability of τ_f. In the proof of the next proposition we also obtain the continuous variation of $\log J_f$ in the $L^1(m)$ norm with $f \in \mathcal{U}$, as explicitly stated in

Proposition 5.4. $\int \log J_{F_n} \, d\mu_{F_n}$ *converges to* $\int \log J_{F_0} \, d\mu_{F_0}$, *when* $n \to \infty$.

Proof. First we write

$$\left|\int \log J_{F_0} d\mu_{F_0} - \int \log J_{F_n} d\mu_{F_n}\right| \leq$$
$$\left|\int (\log J_{F_n} - \log J_{F_0})\rho_n dm\right| + \left|\int (\rho_n - \rho_0) \log J_{F_0} dm\right|.$$

It follows from Proposition 3.1, Proposition 3.3 and Lemma 5.3 that if we take $\varphi_n = \rho_n$ and $\psi = \log J_{F_0}$ then these functions are in the conditions of Lemma 5.1. Thus, it is enough to show that

$$\int \left|\log J_{F_n} - \log J_{F_0}\right| dm \to 0, \quad \text{when } n \to \infty. \tag{5.4}$$

Take any $\varepsilon > 0$. Since $\tau_0 \in L^1(m)$, there is $N \geq 1$ such that

$$\int_{\{\tau_0 > N\}} \tau_0 dm < \varepsilon. \tag{5.5}$$

We then write

$$\int |\log J_{F_n} - \log J_{F_0}|dm = \tag{5.6}$$

$$\int_{\{\tau_n > N\}} |\log J_{F_n} - \log J_{F_0}|dm + \int_{\{\tau_n \leq N\}} |\log J_{F_n} - \log J_{F_0}|dm. \tag{5.7}$$

Let us start by controlling the first in this last sum. Using Lemma 5.3 we obtain

$$\int_{\{\tau_n > N\}} |\log J_{F_n} - \log J_{F_0}|dm \leq C\left(\int_{\{\tau_n > N\}} \tau_n dm + \int_{\{\tau_n > N\}} \tau_0 dm\right). \tag{5.8}$$

One has

$$\mathbf{1}_{\{\tau_n > N\}}\tau_n \leq \mathbf{1}_{\{\tau_0 > N\}}\tau_0 + |\mathbf{1}_{\{\tau_n > N\}} - \mathbf{1}_{\{\tau_0 > N\}}|\tau_0 + \mathbf{1}_{\{\tau_n > N\}}|\tau_n - \tau_0|. \tag{5.9}$$

Choosing n sufficiently large, we have

$$\int \mathbf{1}_{\{\tau_n > N\}}|\tau_n - \tau_0|dm \leq \int |\tau_n - \tau_0|dm < \varepsilon. \tag{5.10}$$

On the other hand, applying Lemma 5.1 to $\varphi_n = \mathbf{1}_{\{\tau_n > N\}}$, for $n \geq 0$, and $\psi = \tau_0$ we also have for large n

$$\int |\mathbf{1}_{\{\tau_n > N\}} - \mathbf{1}_{\{\tau_0 > N\}}|\, \tau_0 dm < \varepsilon. \tag{5.11}$$

It follows from (5.5), (5.9), (5.10) and (5.11) that for large n

$$\int_{\{\tau_n > N\}} \tau_n dm < 3\varepsilon. \tag{5.12}$$

Also from (5.5) and (5.11)

$$\int_{\{\tau_n > N\}} \tau_0 dm \leq \int |\mathbf{1}_{\{\tau_n > N\}} - \mathbf{1}_{\{\tau_0 > N\}}|\, \tau_0 dm + \int_{\{\tau_0 > N\}} \tau_0 dm < 2\varepsilon. \tag{5.13}$$

Hence, from (5.8), (5.12) and (5.13) we deduce that for large n

$$\int_{\{\tau_n > N\}} |\log J_{F_n} - \log J_F|dm < 5C\varepsilon. \tag{5.14}$$

Let us now estimate the second term in (5.6). Letting $C > 0$ be the constant given by Lemma 5.3, take $\delta > 0$ such that

$$\int_B C(N + \tau_0)dm < \varepsilon, \quad \text{whenever}\, m(B) < \delta. \tag{5.15}$$

For each $n \in \mathbb{N}$ define

$$A_n = \big\{x \in \Delta \colon \tau_n(x) = \tau_0(x)\big\}.$$

Since τ_n takes only integer values, we have by (u_1)

$$m(\Delta \setminus A_n) \le \delta, \quad \text{for large } n. \tag{5.16}$$

Observe that for each $x \in A_n$ we have $F_n(x) = f_n^{\tau_0(x)}(x)$. Thus we may write

$$\int_{\{\tau_n \le N\}} |\log J_{F_n} - \log J_{F_0}| dm \le$$

$$\int_{A_n \cap \{\tau_n \le N\}} |\log J_{f_n^{\tau_0}} - \log J_{f_0^{\tau_0}}| dm + \int_{\{\tau_n \le N\} \setminus A_n} |\log J_{F_n} - \log J_{F_0}| dm.$$

Note that by (i_2) we have $J_{f_n^{\tau_0}} \ge 1$ for every $n \ge 0$. Hence, the first integral in the last sum can be made arbitrarily small if we take n sufficiently large. On the other hand, we have by Lemma 5.3

$$\int_{\{\tau_n \le N\} \setminus A_n} |\log J_{F_n} - \log J_{F_0}| dm \le \int_{\Delta \setminus A_n} C(N + \tau_0) dm$$

It follows from (5.15) and (5.16) that this last quantity can be made smaller than $\varepsilon > 0$, as long as n is take sufficiently large. □

6. FAMILIES OF NON-UNIFORMLY EXPANDING MAPS

Finally we prove Theorem C. We just have to check that a family $\mathcal{N}$ as in the statement of Theorem C satisfies the hypotheses of Theorem B. For that we will use results from Ref. (2). It is proved at the end of Section 5 in Ref. (2) that our condition (u_2) and the following two conditions hold for each $f_0 \in \mathcal{N}$:

(v_1) Given $N \in \mathbb{Z}^+$ and $\epsilon > 0$, there is $\delta > 0$ such that for $1 \le j \le N$ and $f \in \mathcal{N}$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad m\big(\{\tau_f = j\} \triangle \{\tau_{f_0} = j\}\big) < \epsilon,$$

where $\triangle$ represents the symmetric difference of two sets.

(v_2) Given $\epsilon > 0$, there are $N \ge 1$ and $\delta > 0$ such that for $f \in \mathcal{N}$

$$\|f - f_0\|_{C^k} < \delta \quad \Rightarrow \quad \Big\| \sum_{j=N}^{\infty} \mathbf{1}_{\{\tau_f > j\}} \Big\|_1 < \epsilon$$

where the L^1-norm is taken with respect to Lebesgue measure.

Thus, we just have to check that conditions (v_1) and (v_2) imply our condition (u_1). Actually, since for each $f \in \mathcal{N}$ we have

$$\tau_f = \sum_{j=0}^{\infty} \mathbf{1}_{\{\tau_f > j\}},$$

then given any $N \in \mathbb{N}$ we may write

$$\left\|\tau_f - \tau_{f_0}\right\| \leq \left\| \sum_{j=N}^{\infty} \mathbf{1}_{\{\tau_f > j\}} \right\|_1 + \left\| \sum_{j=0}^{N-1} \mathbf{1}_{\{\tau_{f_0} > j\}} - \sum_{j=0}^{N-1} \mathbf{1}_{\{\tau_f > j\}} \right\|_1 + \left\| \sum_{j=N}^{\infty} \mathbf{1}_{\{\tau_{f_0} > j\}} \right\|_1$$

Using (v_1) and (v_2) we can easily make all the three terms in the sum above arbitrarily small, as long as we take N sufficiently large and f sufficiently close to f_0. This gives (u_1).

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